Discussion Problems

37. Let $n = 1, 2, 3, \ldots$. Discuss how the observations $D^n x^{n-1} = 0$ and $D^n x^n = n!$ can be used to find the general solutions of the given differential equations.

(a) $y'' = 0$ **(b)** $y''' = 0$ **(c)** $y^{(4)} = 0$

(d) $y'' = 2$ **(e)** $y''' = 6$ **(f)** $y^{(4)} = 24$

38. Suppose that $y_1 = e^x$ and $y_2 = e^{-x}$ are two solutions of a homogeneous linear differential equation. Explain why $y_3 = \cosh x$ and $y_4 = \sinh x$ are also solutions of the equation.

39. **(a)** Verify that $y_1 = x^3$ and $y_2 = |x|^3$ are linearly independent solutions of the differential equation $x^2y'' - 4xy' + 6y = 0$ on the interval $(-\infty, \infty)$.

(b) Show that $W(y_1, y_2) = 0$ for every real number x. Does this result violate Theorem 4.1.3? Explain.

(c) Verify that $Y_1 = x^3$ and $Y_2 = x^2$ are also linearly independent solutions of the differential equation in part (a) on the interval $(-\infty, \infty)$.

(d) Find a solution of the differential equation satisfying $y(0) = 0$, $y'(0) = 0$.

(e) By the superposition principle, Theorem 4.1.2, both linear combinations $y = c_1y_1 + c_2y_2$ and $Y = c_1Y_1 + c_2Y_2$ are solutions of the differential equation. Discuss whether one, both, or neither of the linear combinations is a general solution of the differential equation on the interval $(-\infty, \infty)$.

40. Is the set of functions $f_1(x) = e^{x+2}$, $f_2(x) = e^{x-3}$ linearly dependent or linearly independent on $(-\infty, \infty)$? Discuss.

41. Suppose $y_1, y_2, \ldots, y_k$ are k linearly independent solutions on $(-\infty, \infty)$ of a homogeneous linear nth-order differential equation with constant coefficients. By Theorem 4.1.2 it follows that $y_{k+1} = 0$ is also a solution of the differential equation. Is the set of solutions $y_1, y_2, \ldots, y_k, y_{k+1}$ linearly dependent or linearly independent on $(-\infty, \infty)$? Discuss.

42. Suppose that $y_1, y_2, \ldots, y_k$ are k nontrivial solutions of a homogeneous linear nth-order differential equation with constant coefficients and that $k = n + 1$. Is the set of solutions $y_1, y_2, \ldots, y_k$ linearly dependent or linearly independent on $(-\infty, \infty)$? Discuss.

4.2 REDUCTION OF ORDER

REVIEW MATERIAL

- Section 2.5 (using a substitution)
- Section 4.1

INTRODUCTION In the preceding section we saw that the general solution of a homogeneous linear second-order differential equation

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0 \tag{1}$$

is a linear combination $y = c_1y_1 + c_2y_2$, where y_1 and y_2 are solutions that constitute a linearly independent set on some interval I. Beginning in the next section, we examine a method for determining these solutions when the coefficients of the differential equation in (1) are constants. This method, which is a straightforward exercise in algebra, breaks down in a few cases and yields only a single solution y_1 of the DE. It turns out that we can construct a second solution y_2 of a homogeneous equation (1) (even when the coefficients in (1) are variable) provided that we know a nontrivial solution y_1 of the DE. The basic idea described in this section is that *equation* (1) *can be reduced to a linear first-order DE by means of a substitution* involving the known solution y_1. A second solution y_2 of (1) is apparent after this first-order differential equation is solved.

REDUCTION OF ORDER Suppose that y_1 denotes a nontrivial solution of (1) and that y_1 is defined on an interval I. We seek a second solution y_2 so that the set consisting of y_1 and y_2 is linearly independent on I. Recall from Section 4.1 that if y_1 and y_2 are linearly independent, then their quotient y_2/y_1 is nonconstant on I—that is, $y_2(x)/y_1(x) = u(x)$ or $y_2(x) = u(x)y_1(x)$. The function $u(x)$ can be found by substituting $y_2(x) = u(x)y_1(x)$ into the given differential equation. This method is called **reduction of order** because we must solve a linear first-order differential equation to find u.

EXAMPLE 1 A Second Solution by Reduction of Order

Given that $y_1 = e^x$ is a solution of $y'' - y = 0$ on the interval $(-\infty, \infty)$, use reduction of order to find a second solution y_2.

SOLUTION If $y = u(x)y_1(x) = u(x)e^x$, then the Product Rule gives

$$y' = ue^x + e^x u', \quad y'' = ue^x + 2e^x u' + e^x u'',$$

and so

$$y'' - y = e^x(u'' + 2u') = 0.$$

Since $e^x \neq 0$, the last equation requires $u'' + 2u' = 0$. If we make the substitution $w = u'$, this linear second-order equation in u becomes $w' + 2w = 0$, which is a linear first-order equation in w. Using the integrating factor e^{2x}, we can write $\frac{d}{dx}[e^{2x}w] = 0$. After integrating, we get $w = c_1e^{-2x}$ or $u' = c_1e^{-2x}$. Integrating again then yields $u = -\frac{1}{2}c_1e^{-2x} + c_2$. Thus

$$y = u(x)e^x = -\frac{c_1}{2}e^{-x} + c_2e^x. \tag{2}$$

By picking $c_2 = 0$ and $c_1 = -2$, we obtain the desired second solution, $y_2 = e^{-x}$. Because $W(e^x, e^{-x}) \neq 0$ for every x, the solutions are linearly independent on $(-\infty, \infty)$. ■

Since we have shown that $y_1 = e^x$ and $y_2 = e^{-x}$ are linearly independent solutions of a linear second-order equation, the expression in (2) is actually the general solution of $y'' - y = 0$ on $(-\infty, \infty)$.

GENERAL CASE Suppose we divide by $a_2(x)$ to put equation (1) in the **standard form**

$$y'' + P(x)y' + Q(x)y = 0, \tag{3}$$

where $P(x)$ and $Q(x)$ are continuous on some interval I. Let us suppose further that $y_1(x)$ is a known solution of (3) on I and that $y_1(x) \neq 0$ for every x in the interval. If we define $y = u(x)y_1(x)$, it follows that

$$y' = uy_1' + y_1u', \quad y'' = uy_1'' + 2y_1'u' + y_1u''$$

$$y'' + Py' + Qy = u[\underbrace{y_1'' + Py_1' + Qy_1}_{\text{zero}}] + y_1u'' + (2y_1' + Py_1)u' = 0.$$

This implies that we must have

$$y_1u'' + (2y_1' + Py_1)u' = 0 \quad \text{or} \quad y_1w' + (2y_1' + Py_1)w = 0, \tag{4}$$

where we have let $w = u'$. Observe that the last equation in (4) is both linear and separable. Separating variables and integrating, we obtain

$$\frac{dw}{w} + 2\frac{y_1'}{y_1}dx + P\,dx = 0$$

$$\ln|wy_1^2| = -\int P\,dx + c \quad \text{or} \quad wy_1^2 = c_1e^{-\int P\,dx}.$$

We solve the last equation for w, use $w = u'$, and integrate again:

$$u = c_1\int \frac{e^{-\int P\,dx}}{y_1^2}\,dx + c_2.$$

By choosing $c_1 = 1$ and $c_2 = 0$, we find from $y = u(x)y_1(x)$ that a second solution of equation (3) is

$$y_2 = y_1(x) \int \frac{e^{-\int P(x)\,dx}}{y_1^2(x)}\,dx. \tag{5}$$

It makes a good review of differentiation to verify that the function $y_2(x)$ defined in (5) satisfies equation (3) and that y_1 and y_2 are linearly independent on any interval on which $y_1(x)$ is not zero.

EXAMPLE 2 A Second Solution by Formula (5)

The function $y_1 = x^2$ is a solution of $x^2y'' - 3xy' + 4y = 0$. Find the general solution of the differential equation on the interval $(0, \infty)$.

SOLUTION From the standard form of the equation,

$$y'' - \frac{3}{x}y' + \frac{4}{x^2}y = 0,$$

we find from (5)

$$y_2 = x^2 \int \frac{e^{3\int dx/x}}{x^4}\,dx \quad \leftarrow e^{3\int dx/x} = e^{\ln x^3} = x^3$$

$$= x^2 \int \frac{dx}{x} = x^2 \ln x.$$

The general solution on the interval $(0, \infty)$ is given by $y = c_1y_1 + c_2y_2$; that is, $y = c_1x^2 + c_2x^2 \ln x$. ■

REMARKS

(*i*) The derivation and use of formula (5) have been illustrated here because this formula appears again in the next section and in Sections 4.7 and 6.2. We use (5) simply to save time in obtaining a desired result. Your instructor will tell you whether you should memorize (5) or whether you should know the first principles of reduction of order.

(*ii*) Reduction of order can be used to find the general solution of a nonhomogeneous equation $a_2(x)y'' + a_1(x)y' + a_0(x)y = g(x)$ whenever a solution y_1 of the associated homogeneous equation is known. See Problems 17–20 in Exercises 4.2.

EXERCISES 4.2

Answers to selected odd-numbered problems begin on page ANS-4.

In Problems 1–16 the indicated function $y_1(x)$ is a solution of the given differential equation. Use reduction of order or formula (5), as instructed, to find a second solution $y_2(x)$.

1. $y'' - 4y' + 4y = 0$; $y_1 = e^{2x}$

2. $y'' + 2y' + y = 0$; $y_1 = xe^{-x}$

3. $y'' + 16y = 0$; $y_1 = \cos 4x$

4. $y'' + 9y = 0$; $y_1 = \sin 3x$

5. $y'' - y = 0$; $y_1 = \cosh x$

6. $y'' - 25y = 0$; $y_1 = e^{5x}$

7. $9y'' - 12y' + 4y = 0$; $y_1 = e^{2x/3}$

8. $6y'' + y' - y = 0$; $y_1 = e^{x/3}$

9. $x^2y'' - 7xy' + 16y = 0$; $y_1 = x^4$

10. $x^2y'' + 2xy' - 6y = 0$; $y_1 = x^2$

11. $xy'' + y' = 0$; $y_1 = \ln x$

12. $4x^2y'' + y = 0$; $y_1 = x^{1/2} \ln x$

13. $x^2y'' - xy' + 2y = 0$; $y_1 = x \sin(\ln x)$

14. $x^2y'' - 3xy' + 5y = 0$; $y_1 = x^2 \cos(\ln x)$

15. $(1 - 2x - x^2)y'' + 2(1 + x)y' - 2y = 0; \quad y_1 = x + 1$

16. $(1 - x^2)y'' + 2xy' = 0; \quad y_1 = 1$

In Problems 17–20 the indicated function $y_1(x)$ is a solution of the associated homogeneous equation. Use the method of reduction of order to find a second solution $y_2(x)$ of the homogeneous equation and a particular solution of the given nonhomogeneous equation.

17. $y'' - 4y = 2; \quad y_1 = e^{-2x}$

18. $y'' + y' = 1; \quad y_1 = 1$

19. $y'' - 3y' + 2y = 5e^{3x}; \quad y_1 = e^x$

20. $y'' - 4y' + 3y = x; \quad y_1 = e^x$

Discussion Problems

21. **(a)** Give a convincing demonstration that the second-order equation $ay'' + by' + cy = 0$, a, b, and c constants, always possesses at least one solution of the form $y_1 = e^{m_1 x}$, m_1 a constant.

(b) Explain why the differential equation in part (a) must then have a second solution either of the form $y_2 = e^{m_2 x}$ or of the form $y_2 = xe^{m_1 x}$, m_1 and m_2 constants.

(c) Reexamine Problems 1–8. Can you explain why the statements in parts (a) and (b) above are not contradicted by the answers to Problems 3–5?

22. Verify that $y_1(x) = x$ is a solution of $xy'' - xy' + y = 0$. Use reduction of order to find a second solution $y_2(x)$ in the form of an infinite series. Conjecture an interval of definition for $y_2(x)$.

Computer Lab Assignments

23. **(a)** Verify that $y_1(x) = e^x$ is a solution of

$$xy'' - (x + 10)y' + 10y = 0.$$

(b) Use (5) to find a second solution $y_2(x)$. Use a CAS to carry out the required integration.

(c) Explain, using Corollary (A) of Theorem 4.1.2, why the second solution can be written compactly as

$$y_2(x) = \sum_{n=0}^{10} \frac{1}{n!} x^n.$$

4.3 HOMOGENEOUS LINEAR EQUATIONS WITH CONSTANT COEFFICIENTS

REVIEW MATERIAL

- Review Problem 27 in Exercises 1.1 and Theorem 4.1.5
- Review the algebra of solving polynomial equations (see the *Student Resource and Solutions Manual*)

INTRODUCTION As a means of motivating the discussion in this section, let us return to first-order differential equations—more specifically, to *homogeneous* linear equations $ay' + by = 0$, where the coefficients $a \neq 0$ and b are constants. This type of equation can be solved either by separation of variables or with the aid of an integrating factor, but there is another solution method, one that uses only algebra. Before illustrating this alternative method, we make one observation: Solving $ay' + by = 0$ for y' yields $y' = ky$, where k is a constant. This observation reveals the nature of the unknown solution y; the only nontrivial elementary function whose derivative is a constant multiple of itself is an exponential function e^{mx}. Now the new solution method: If we substitute $y = e^{mx}$ and $y' = me^{mx}$ into $ay' + by = 0$, we get

$$ame^{mx} + be^{mx} = 0 \qquad \text{or} \qquad e^{mx}(am + b) = 0.$$

Since e^{mx} is never zero for real values of x, the last equation is satisfied only when m is a solution or root of the first-degree polynomial equation $am + b = 0$. For this single value of m, $y = e^{mx}$ is a solution of the DE. To illustrate, consider the constant-coefficient equation $2y' + 5y = 0$. It is not necessary to go through the differentiation and substitution of $y = e^{mx}$ into the DE; we merely have to form the equation $2m + 5 = 0$ and solve it for m. From $m = -\frac{5}{2}$ we conclude that $y = e^{-5x/2}$ is a solution of $2y' + 5y = 0$, and its general solution on the interval $(-\infty, \infty)$ is $y = c_1 e^{-5x/2}$.

In this section we will see that the foregoing procedure can produce exponential solutions for homogeneous linear higher-order DEs,

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_2 y'' + a_1 y' + a_0 y = 0, \tag{1}$$

where the coefficients a_i, $i = 0, 1, \ldots, n$ are real constants and $a_n \neq 0$.

AUXILIARY EQUATION We begin by considering the special case of the second-order equation

$$ay'' + by' + cy = 0, \tag{2}$$

where a, b, and c are constants. If we try to find a solution of the form $y = e^{mx}$, then after substitution of $y' = me^{mx}$ and $y'' = m^2e^{mx}$, equation (2) becomes

$$am^2e^{mx} + bme^{mx} + ce^{mx} = 0 \quad \text{or} \quad e^{mx}(am^2 + bm + c) = 0.$$

As in the introduction we argue that because $e^{mx} \neq 0$ for all x, it is apparent that the only way $y = e^{mx}$ can satisfy the differential equation (2) is when m is chosen as a root of the quadratic equation

$$am^2 + bm + c = 0. \tag{3}$$

This last equation is called the **auxiliary equation** of the differential equation (2). Since the two roots of (3) are $m_1 = (-b + \sqrt{b^2 - 4ac})/2a$ and $m_2 = (-b - \sqrt{b^2 - 4ac})/2a$, there will be three forms of the general solution of (2) corresponding to the three cases:

- m_1 and m_2 real and distinct ($b^2 - 4ac > 0$),
- m_1 and m_2 real and equal ($b^2 - 4ac = 0$), and
- m_1 and m_2 conjugate complex numbers ($b^2 - 4ac < 0$).

We discuss each of these cases in turn.

CASE I: DISTINCT REAL ROOTS Under the assumption that the auxiliary equation (3) has two unequal real roots m_1 and m_2, we find two solutions, $y_1 = e^{m_1x}$ and $y_2 = e^{m_2x}$. We see that these functions are linearly independent on $(-\infty, \infty)$ and hence form a fundamental set. It follows that the general solution of (2) on this interval is

$$y = c_1e^{m_1x} + c_2e^{m_2x}. \tag{4}$$

CASE II: REPEATED REAL ROOTS When $m_1 = m_2$, we necessarily obtain only one exponential solution, $y_1 = e^{m_1x}$. From the quadratic formula we find that $m_1 = -b/2a$ since the only way to have $m_1 = m_2$ is to have $b^2 - 4ac = 0$. It follows from (5) in Section 4.2 that a second solution of the equation is

$$y_2 = e^{m_1x}\int \frac{e^{2m_1x}}{e^{2m_1x}}\,dx = e^{m_1x}\int dx = xe^{m_1x}. \tag{5}$$

In (5) we have used the fact that $-b/a = 2m_1$. The general solution is then

$$y = c_1e^{m_1x} + c_2xe^{m_1x}. \tag{6}$$

CASE III: CONJUGATE COMPLEX ROOTS If m_1 and m_2 are complex, then we can write $m_1 = \alpha + i\beta$ and $m_2 = \alpha - i\beta$, where α and $\beta > 0$ are real and $i^2 = -1$. Formally, there is no difference between this case and Case I, and hence

$$y = C_1e^{(\alpha+i\beta)x} + C_2e^{(\alpha-i\beta)x}.$$

However, in practice we prefer to work with real functions instead of complex exponentials. To this end we use **Euler's formula:**

$$e^{i\theta} = \cos\theta + i\sin\theta,$$

where θ is any real number.* It follows from this formula that

$$e^{i\beta x} = \cos\beta x + i\sin\beta x \quad \text{and} \quad e^{-i\beta x} = \cos\beta x - i\sin\beta x, \tag{7}$$

*A formal derivation of Euler's formula can be obtained from the Maclaurin series $e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!}$ by substituting $x = i\theta$, using $i^2 = -1$, $i^3 = -i$, . . . , and then separating the series into real and imaginary parts. The plausibility thus established, we can adopt $\cos\theta + i\sin\theta$ as the *definition* of $e^{i\theta}$.

where we have used $\cos(-\beta x) = \cos \beta x$ and $\sin(-\beta x) = -\sin \beta x$. Note that by first adding and then subtracting the two equations in (7), we obtain, respectively,

$$e^{i\beta x} + e^{-i\beta x} = 2\cos \beta x \qquad \text{and} \qquad e^{i\beta x} - e^{-i\beta x} = 2i \sin \beta x.$$

Since $y = C_1 e^{(\alpha + i\beta)x} + C_2 e^{(\alpha - i\beta)x}$ is a solution of (2) for any choice of the constants C_1 and C_2, the choices $C_1 = C_2 = 1$ and $C_1 = 1$, $C_2 = -1$ give, in turn, two solutions:

$$y_1 = e^{(\alpha + i\beta)x} + e^{(\alpha - i\beta)x} \qquad \text{and} \qquad y_2 = e^{(\alpha + i\beta)x} - e^{(\alpha - i\beta)x}.$$

But $$y_1 = e^{\alpha x}(e^{i\beta x} + e^{-i\beta x}) = 2e^{\alpha x}\cos \beta x$$

and $$y_2 = e^{\alpha x}(e^{i\beta x} - e^{-i\beta x}) = 2ie^{\alpha x}\sin \beta x.$$

Hence from Corollary (A) of Theorem 4.1.2 the last two results show that $e^{\alpha x}\cos \beta x$ and $e^{\alpha x}\sin \beta x$ are *real* solutions of (2). Moreover, these solutions form a fundamental set on $(-\infty, \infty)$. Consequently, the general solution is

$$y = c_1 e^{\alpha x}\cos \beta x + c_2 e^{\alpha x}\sin \beta x = e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x). \tag{8}$$

EXAMPLE 1 **Second-Order DEs**

Solve the following differential equations.

(a) $2y'' - 5y' - 3y = 0$ **(b)** $y'' - 10y' + 25y = 0$ **(c)** $y'' + 4y' + 7y = 0$

SOLUTION We give the auxiliary equations, the roots, and the corresponding general solutions.

(a) $2m^2 - 5m - 3 = (2m + 1)(m - 3) = 0, \quad m_1 = -\frac{1}{2}, m_2 = 3$

From (4), $y = c_1 e^{-x/2} + c_2 e^{3x}$.

(b) $m^2 - 10m + 25 = (m - 5)^2 = 0, \quad m_1 = m_2 = 5$

From (6), $y = c_1 e^{5x} + c_2 x e^{5x}$.

(c) $m^2 + 4m + 7 = 0, \; m_1 = -2 + \sqrt{3}i, \quad m_2 = -2 - \sqrt{3}i$

From (8) with $\alpha = -2$, $\beta = \sqrt{3}$, $y = e^{-2x}\left(c_1 \cos \sqrt{3}x + c_2 \sin \sqrt{3}x\right)$. ■

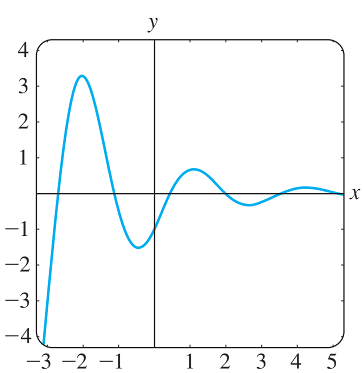

FIGURE 4.3.1 Solution curve of IVP in Example 2

EXAMPLE 2 **An Initial-Value Problem**

Solve $4y'' + 4y' + 17y = 0$, $y(0) = -1$, $y'(0) = 2$.

SOLUTION By the quadratic formula we find that the roots of the auxiliary equation $4m^2 + 4m + 17 = 0$ are $m_1 = -\frac{1}{2} + 2i$ and $m_2 = -\frac{1}{2} - 2i$. Thus from (8) we have $y = e^{-x/2}(c_1 \cos 2x + c_2 \sin 2x)$. Applying the condition $y(0) = -1$, we see from $e^0(c_1 \cos 0 + c_2 \sin 0) = -1$ that $c_1 = -1$. Differentiating $y = e^{-x/2}(-\cos 2x + c_2 \sin 2x)$ and then using $y'(0) = 2$ gives $2c_2 + \frac{1}{2} = 2$ or $c_2 = \frac{3}{4}$. Hence the solution of the IVP is $y = e^{-x/2}(-\cos 2x + \frac{3}{4}\sin 2x)$. In Figure 4.3.1 we see that the solution is oscillatory, but $y \to 0$ as $x \to \infty$ and $|y| \to \infty$ as $x \to -\infty$. ■

TWO EQUATIONS WORTH KNOWING The two differential equations

$$y'' + k^2 y = 0 \qquad \text{and} \qquad y'' - k^2 y = 0,$$

where k is real, are important in applied mathematics. For $y'' + k^2y = 0$ the auxiliary equation $m^2 + k^2 = 0$ has imaginary roots $m_1 = ki$ and $m_2 = -ki$. With $\alpha = 0$ and $\beta = k$ in (8) the general solution of the DE is seen to be

$$y = c_1 \cos kx + c_2 \sin kx. \tag{9}$$

On the other hand, the auxiliary equation $m^2 - k^2 = 0$ for $y'' - k^2y = 0$ has distinct real roots $m_1 = k$ and $m_2 = -k$, and so by (4) the general solution of the DE is

$$y = c_1e^{kx} + c_2e^{-kx}. \tag{10}$$

Notice that if we choose $c_1 = c_2 = \frac{1}{2}$ and $c_1 = \frac{1}{2}$, $c_2 = -\frac{1}{2}$ in (10), we get the particular solutions $y = \frac{1}{2}(e^{kx} + e^{-kx}) = \cosh kx$ and $y = \frac{1}{2}(e^{kx} - e^{-kx}) = \sinh kx$. Since $\cosh kx$ and $\sinh kx$ are linearly independent on any interval of the x-axis, an alternative form for the general solution of $y'' - k^2y = 0$ is

$$y = c_1 \cosh kx + c_2 \sinh kx. \tag{11}$$

See Problems 41 and 42 in Exercises 4.3.

HIGHER-ORDER EQUATIONS In general, to solve an nth-order differential equation (1), where the a_i, $i = 0, 1, \ldots, n$ are real constants, we must solve an nth-degree polynomial equation

$$a_nm^n + a_{n-1}m^{n-1} + \cdots + a_2m^2 + a_1m + a_0 = 0. \tag{12}$$

If all the roots of (12) are real and distinct, then the general solution of (1) is

$$y = c_1e^{m_1x} + c_2e^{m_2x} + \cdots + c_ne^{m_nx}.$$

It is somewhat harder to summarize the analogues of Cases II and III because the roots of an auxiliary equation of degree greater than two can occur in many combinations. For example, a fifth-degree equation could have five distinct real roots, or three distinct real and two complex roots, or one real and four complex roots, or five real but equal roots, or five real roots but two of them equal, and so on. When m_1 is a root of multiplicity k of an nth-degree auxiliary equation (that is, k roots are equal to m_1), it can be shown that the linearly independent solutions are

$$e^{m_1x}, \quad xe^{m_1x}, \quad x^2e^{m_1x}, \ldots, \quad x^{k-1}e^{m_1x}$$

and the general solution must contain the linear combination

$$c_1e^{m_1x} + c_2xe^{m_1x} + c_3x^2e^{m_1x} + \cdots + c_kx^{k-1}e^{m_1x}.$$

Finally, it should be remembered that when the coefficients are real, complex roots of an auxiliary equation always appear in conjugate pairs. Thus, for example, a cubic polynomial equation can have at most two complex roots.

EXAMPLE 3 Third-Order DE

Solve $y''' + 3y'' - 4y = 0$.

SOLUTION It should be apparent from inspection of $m^3 + 3m^2 - 4 = 0$ that one root is $m_1 = 1$, so $m - 1$ is a factor of $m^3 + 3m^2 - 4$. By division we find

$$m^3 + 3m^2 - 4 = (m - 1)(m^2 + 4m + 4) = (m - 1)(m + 2)^2,$$

so the other roots are $m_2 = m_3 = -2$. Thus the general solution of the DE is $y = c_1e^x + c_2e^{-2x} + c_3xe^{-2x}$. ■

EXAMPLE 4 Fourth-Order DE

Solve $\dfrac{d^4y}{dx^4} + 2\dfrac{d^2y}{dx^2} + y = 0.$

SOLUTION The auxiliary equation $m^4 + 2m^2 + 1 = (m^2 + 1)^2 = 0$ has roots $m_1 = m_3 = i$ and $m_2 = m_4 = -i$. Thus from Case II the solution is

$$y = C_1e^{ix} + C_2e^{-ix} + C_3xe^{ix} + C_4xe^{-ix}.$$

By Euler's formula the grouping $C_1e^{ix} + C_2e^{-ix}$ can be rewritten as

$$c_1 \cos x + c_2 \sin x$$

after a relabeling of constants. Similarly, $x(C_3e^{ix} + C_4e^{-ix})$ can be expressed as $x(c_3 \cos x + c_4 \sin x)$. Hence the general solution is

$$y = c_1 \cos x + c_2 \sin x + c_3x \cos x + c_4x \sin x.$$ ■

Example 4 illustrates a special case when the auxiliary equation has repeated complex roots. In general, if $m_1 = \alpha + i\beta$, $\beta > 0$ is a complex root of multiplicity k of an auxiliary equation with real coefficients, then its conjugate $m_2 = \alpha - i\beta$ is also a root of multiplicity k. From the $2k$ complex-valued solutions

$$e^{(\alpha+i\beta)x}, \quad xe^{(\alpha+i\beta)x}, \quad x^2e^{(\alpha+i\beta)x}, \quad \ldots, \quad x^{k-1}e^{(\alpha+i\beta)x},$$
$$e^{(\alpha-i\beta)x}, \quad xe^{(\alpha-i\beta)x}, \quad x^2e^{(\alpha-i\beta)x}, \quad \ldots, \quad x^{k-1}e^{(\alpha-i\beta)x},$$

we conclude, with the aid of Euler's formula, that the general solution of the corresponding differential equation must then contain a linear combination of the $2k$ real linearly independent solutions

$$e^{\alpha x}\cos \beta x, \quad xe^{\alpha x}\cos \beta x, \quad x^2e^{\alpha x}\cos \beta x, \quad \ldots, \quad x^{k-1}e^{\alpha x}\cos \beta x,$$
$$e^{\alpha x}\sin \beta x, \quad xe^{\alpha x}\sin \beta x, \quad x^2e^{\alpha x}\sin \beta x, \quad \ldots, \quad x^{k-1}e^{\alpha x}\sin \beta x.$$

In Example 4 we identify $k = 2$, $\alpha = 0$, and $\beta = 1$.

Of course the most difficult aspect of solving constant-coefficient differential equations is finding roots of auxiliary equations of degree greater than two. For example, to solve $3y''' + 5y'' + 10y' - 4y = 0$, we must solve $3m^3 + 5m^2 + 10m - 4 = 0$. Something we can try is to test the auxiliary equation for rational roots. Recall that if $m_1 = p/q$ is a rational root (expressed in lowest terms) of an auxiliary equation $a_nm^n + \cdots + a_1m + a_0 = 0$ with integer coefficients, then p is a factor of a_0 and q is a factor of a_n. For our specific cubic auxiliary equation, all the factors of $a_0 = -4$ and $a_n = 3$ are p: $\pm1, \pm2, \pm4$ and q: $\pm1, \pm3$, so the possible rational roots are p/q: $\pm1, \pm2, \pm4, \pm\frac{1}{3}, \pm\frac{2}{3}, \pm\frac{4}{3}$ Each of these numbers can then be tested—say, by synthetic division. In this way we discover both the root $m_1 = \frac{1}{3}$ and the factorization

$$3m^3 + 5m^2 + 10m - 4 = \left(m - \tfrac{1}{3}\right)(3m^2 + 6m + 12).$$

The quadratic formula then yields the remaining roots $m_2 = -1 + \sqrt{3}i$ and $m_3 = -1 - \sqrt{3}i$. Therefore the general solution of $3y''' + 5y'' + 10y' - 4y = 0$ is $y = c_1e^{x/3} + e^{-x}(c_2 \cos \sqrt{3}x + c_3 \sin \sqrt{3}x)$.

USE OF COMPUTERS Finding roots or approximation of roots of auxiliary equations is a routine problem with an appropriate calculator or computer software. Polynomial equations (in one variable) of degree less than five can be solved by means of algebraic formulas using the *solve* commands in *Mathematica* and *Maple*. For auxiliary equations of degree five or greater it might be necessary to resort to numerical commands such as **NSolve** and **FindRoot** in *Mathematica*. Because of their capability of solving polynomial equations, it is not surprising that these computer algebra

■ There is more on this in the ***SRSM***.

systems are also able, by means of their *dsolve* commands, to provide explicit solutions of homogeneous linear constant-coefficient differential equations.

In the classic text *Differential Equations* by Ralph Palmer Agnew* (used by the author as a student) the following statement is made:

> *It is not reasonable to expect students in this course to have computing skill and equipment necessary for efficient solving of equations such as*

$$4.317\frac{d^4y}{dx^4} + 2.179\frac{d^3y}{dx^3} + 1.416\frac{d^2y}{dx^2} + 1.295\frac{dy}{dx} + 3.169y = 0. \tag{13}$$

Although it is debatable whether computing skills have improved in the intervening years, it is a certainty that technology has. If one has access to a computer algebra system, equation (13) could now be considered reasonable. After simplification and some relabeling of output, *Mathematica* yields the (approximate) general solution

$$\begin{aligned} y &= c_1e^{-0.728852x}\cos(0.618605x) + c_2e^{-0.728852x}\sin(0.618605x) \\ &\quad + c_3e^{0.476478x}\cos(0.759081x) + c_4e^{0.476478x}\sin(0.759081x). \end{aligned}$$

Finally, if we are faced with an initial-value problem consisting of, say, a fourth-order equation, then to fit the general solution of the DE to the four initial conditions, we must solve four linear equations in four unknowns (the c_1, c_2, c_3, c_4 in the general solution). Using a CAS to solve the system can save lots of time. See Problems 59 and 60 in Exercises 4.3 and Problem 35 in Chapter 4 in Review.

*McGraw-Hill, New York, 1960.

EXERCISES 4.3

Answers to selected odd-numbered problems begin on page ANS-4.

In Problems 1–14 find the general solution of the given second-order differential equation.

1. $4y'' + y' = 0$

2. $y'' - 36y = 0$

3. $y'' - y' - 6y = 0$

4. $y'' - 3y' + 2y = 0$

5. $y'' + 8y' + 16y = 0$

6. $y'' - 10y' + 25y = 0$

7. $12y'' - 5y' - 2y = 0$

8. $y'' + 4y' - y = 0$

9. $y'' + 9y = 0$

10. $3y'' + y = 0$

11. $y'' - 4y' + 5y = 0$

12. $2y'' + 2y' + y = 0$

13. $3y'' + 2y' + y = 0$

14. $2y'' - 3y' + 4y = 0$

In Problems 15–28 find the general solution of the given higher-order differential equation.

15. $y''' - 4y'' - 5y' = 0$

16. $y''' - y = 0$

17. $y''' - 5y'' + 3y' + 9y = 0$

18. $y''' + 3y'' - 4y' - 12y = 0$

19. $\dfrac{d^3u}{dt^3} + \dfrac{d^2u}{dt^2} - 2u = 0$

20. $\dfrac{d^3x}{dt^3} - \dfrac{d^2x}{dt^2} - 4x = 0$

21. $y''' + 3y'' + 3y' + y = 0$

22. $y''' - 6y'' + 12y' - 8y = 0$

23. $y^{(4)} + y''' + y'' = 0$

24. $y^{(4)} - 2y'' + y = 0$

25. $16\dfrac{d^4y}{dx^4} + 24\dfrac{d^2y}{dx^2} + 9y = 0$

26. $\dfrac{d^4y}{dx^4} - 7\dfrac{d^2y}{dx^2} - 18y = 0$

27. $\dfrac{d^5u}{dr^5} + 5\dfrac{d^4u}{dr^4} - 2\dfrac{d^3u}{dr^3} - 10\dfrac{d^2u}{dr^2} + \dfrac{du}{dr} + 5u = 0$

28. $2\dfrac{d^5x}{ds^5} - 7\dfrac{d^4x}{ds^4} + 12\dfrac{d^3x}{ds^3} + 8\dfrac{d^2x}{ds^2} = 0$

In Problems 29–36 solve the given initial-value problem.

29. $y'' + 16y = 0,\quad y(0) = 2,\ y'(0) = -2$

30. $\dfrac{d^2y}{d\theta^2} + y = 0,\quad y\left(\dfrac{\pi}{3}\right) = 0,\ y'\left(\dfrac{\pi}{3}\right) = 2$

31. $\dfrac{d^2y}{dt^2} - 4\dfrac{dy}{dt} - 5y = 0, \quad y(1) = 0, y'(1) = 2$

32. $4y'' - 4y' - 3y = 0, \quad y(0) = 1, y'(0) = 5$

33. $y'' + y' + 2y = 0, \quad y(0) = y'(0) = 0$

34. $y'' - 2y' + y = 0, \quad y(0) = 5, y'(0) = 10$

35. $y''' + 12y'' + 36y' = 0, \quad y(0) = 0, y'(0) = 1, y''(0) = -7$

36. $y''' + 2y'' - 5y' - 6y = 0, \quad y(0) = y'(0) = 0, \ y''(0) = 1$

In Problems 37–40 solve the given boundary-value problem.

37. $y'' - 10y' + 25y = 0, \quad y(0) = 1, y(1) = 0$

38. $y'' + 4y = 0, \quad y(0) = 0, y(\pi) = 0$

39. $y'' + y = 0, \quad y'(0) = 0, y'\left(\dfrac{\pi}{2}\right) = 0$

40. $y'' - 2y' + 2y = 0, \quad y(0) = 1, y(\pi) = 1$

In Problems 41 and 42 solve the given problem first using the form of the general solution given in (10). Solve again, this time using the form given in (11).

41. $y'' - 3y = 0, \quad y(0) = 1, y'(0) = 5$

42. $y'' - y = 0, \quad y(0) = 1, y'(1) = 0$

In Problems 43–48 each figure represents the graph of a particular solution of one of the following differential equations:

(a) $y'' - 3y' - 4y = 0$ **(b)** $y'' + 4y = 0$

(c) $y'' + 2y' + y = 0$ **(d)** $y'' + y = 0$

(e) $y'' + 2y' + 2y = 0$ **(f)** $y'' - 3y' + 2y = 0$

Match a solution curve with one of the differential equations. Explain your reasoning.

43.

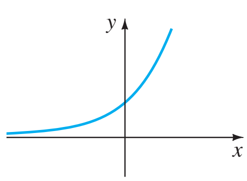

FIGURE 4.3.2 Graph for Problem 43

44.

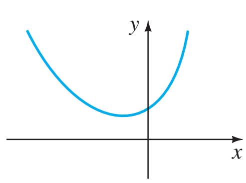

FIGURE 4.3.3 Graph for Problem 44

45.

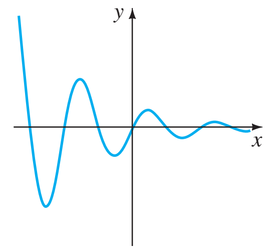

FIGURE 4.3.4 Graph for Problem 45

46.

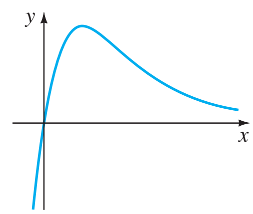

FIGURE 4.3.5 Graph for Problem 46

47.

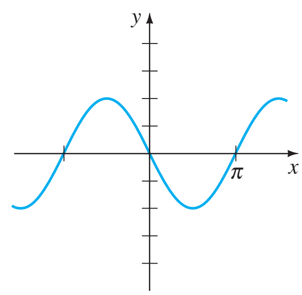

FIGURE 4.3.6 Graph for Problem 47

48.

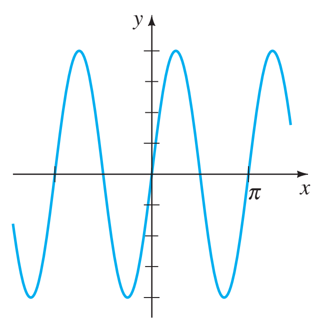

FIGURE 4.3.7 Graph for Problem 48

Discussion Problems

49. The roots of a cubic auxiliary equation are $m_1 = 4$ and $m_2 = m_3 = -5$. What is the corresponding homogeneous linear differential equation? Discuss: Is your answer unique?

50. Two roots of a cubic auxiliary equation with real coefficients are $m_1 = -\frac{1}{2}$ and $m_2 = 3 + i$. What is the corresponding homogeneous linear differential equation?